

NULL-CONTROLLABILITY OF THE KURAMOTO-SIVASHINSKY EQUATION ON STAR-SHAPED TREES

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ABSTRACT. In this paper we treat controllability properties for the linear Kuramoto-Sivashinsky equation on a network with two types of boundary conditions. More precisely, the equation is considered on a star-shaped tree with Dirichlet and Neumann boundary conditions. By using the moment theory we can derive null-controllability properties with boundary controls acting on the external vertices of the tree. In particular, the controllability holds if the *anti-diffusion* parameter of the involved equation does not belong to a critical countable set of real numbers. We point out that the critical set for which the null-controllability fails differs from the first model to the second one.

Key words: Kuramoto-Sivashinsky equation, null-controllability, star-shaped trees, the problem of moments.

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1. INTRODUCTION

In this paper, we consider two control problems on the same simple network formed by the edges of a tree. The problem we address here enter in the framework of quantum graphs. The name quantum graph is used for a graph considered as a one-dimensional singular variety and equipped with a differential operator. Those quantum graphs are metric spaces which can be written as the union of finitely many intervals, which are compact or $[0, \infty)$ and any two of these intervals are either disjoint or intersect only at one of their endpoints.

Our main goal is to study boundary null-controllability properties for the Kuramoto-Sivashinsky (KS) equation

$$(1) \quad y_t + \lambda y_{xx} + y_{xxx} = 0,$$

on a star-shaped tree denoted Γ . More precisely, Γ is a simplified topological graph with $N \geq 2$ edges of the same given length $L > 0$ and $N + 1$ vertices. Besides, all edges intersect at a unique endpoint which is the interior vertex of the graph. The mathematical formulation of the control problems that we address on Γ stands for a system of N -KS equations on the interval $(0, L)$ coupled through the left endpoint

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$x = 0$ as follows

$$(2) \quad \begin{cases} y_t^k + \lambda y_{xx}^k + y_{xxxx}^k = 0, & (t, x) \in (0, T) \times (0, L) \\ y^i(t, 0) = y^j(t, 0), & t \in (0, T), \quad i, j \in \{1, \dots, N\} \\ \sum_{k=1}^N y_x^k(t, 0) = 0, & t \in (0, T) \\ y_{xx}^i(t, 0) = y_{xx}^j(t, 0), & t \in (0, T), \quad i, j \in \{1, \dots, N\} \\ \sum_{k=1}^N y_{xxx}^k(t, 0) = 0, & t \in (0, T) \\ y^k(0, x) = y_0^k(x), & x \in (0, L). \end{cases}$$

For system (2) we study two types of boundary control conditions:

$$(I) : \quad \begin{cases} y^k(t, L) = 0, \\ y_x^k(t, L) = u^k(t), \quad k \in \{1, \dots, N\}, \end{cases}$$

respectively

$$(II) : \quad \begin{cases} y_x^k(t, L) = a^k(t), \\ y_{xxx}^k(t, L) = b^k(t), \quad k \in \{1, \dots, N\}. \end{cases}$$

Next in the paper we will refer to (2)-(I) for system (2) subject to the boundary conditions (I) and to (2)-(II) for system (2) with boundary conditions (II).

In system (2), λ is a positive constant, the functions $y^k = y^k(t, x)$ are real-valued for any $k \in \{1, \dots, N\}$, t denotes the time variable, x denotes the space variable and the subscripts for both t and x indicate partial differentiation with respect to each one. The boundary functions u^k , a^k and b^k are considered as control inputs acting on the external nodes. In model (II) we impose two controls to act on the same vertex whereas in model (I) we only require one control. Our purpose is to see whether we can force the solutions of system (2) to have certain properties by choosing appropriate control inputs. The focus here is on the following null-controllability issue:

Given any finite time $T > 0$ and any initial state $y_0 = (y_0^k)_{k=1, N}$, can we find proper control inputs in (I) or (II) ($u = (u^k)_{k=1, N}$ and $a = (a^k)_{k=1, N}$, $b = (b^k)_{k=1, N}$, respectively) to lead the solution of system (2) to the zero state, i.e.,

$$(3) \quad y^k(T, x) = 0, \quad \text{for any } x \in (0, L), \quad k \in \{1, \dots, N\}?$$

For parabolic control problems, in general, it is not possible to steer the system to an arbitrary prescribed state. Thus, we do not expect the exact controllability to be true neither for the KS control system. Our motivation for studying such control systems goes back to the quasilinear KS equation

$$(4) \quad y_t + \lambda y_{xx} + y_{xxxx} + yy_x = 0,$$

which was derived independently by Kuramoto and Tsuzuki in [14, 15] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky in [16, 17] as a model for plane flame propagation. The real positive number λ in (4) is called the anti-diffusion parameter. This nonlinear partial differential equation also describes incipient instabilities in a variety of physical and chemical systems (see, for instance, [9] and [13]).

The linear control problem on the interval $(0, 1)$ has been first studied in [4] considering Dirichlet boundary conditions. By using the moment theory developed

by Fattorini and Russell in [12], it was proved that this system is null controllable if two controls act only on the left endpoint of the interval. If one control is removed the system is controllable if and only if the anti-diffusion parameter λ does not belong to the following countable set of critical values:

$$(5) \quad \mathcal{N}_0 := \left\{ \pi^2(m^2 + n^2) : (m, n) \in \mathbb{N}^2, 0 \leq m < n, m \text{ and } n \text{ have the same parity} \right\}.$$

More precisely, there exists a finite-dimensional space of initial conditions that cannot be driven to zero with only one control. We point out that the results in [4] could be extended to intervals of any length $L > 0$ by re-scaling the set \mathcal{N}_0 in terms of L accordingly.

Later on, the boundary controllability to the trajectories of (4) was proved in [6] when two controls act on the left Dirichlet boundary conditions. We also refer to [2, 5, 7, 8] for related problems and results on the subject.

To our knowledge, the study of the controllability properties of KS systems in the context of quantum graphs has not been yet addressed in the literature neither for the linear problem (1). At this respect, the program of this work was carried out for a choice of classical boundary conditions and aims to establish as a fact that the models under consideration inherit the interesting controllability properties initially observed for the KS equation posed on a bounded interval.

In order to present our main results, we introduce the following countable sets

$$\begin{aligned} \mathcal{N}_1 &:= \left\{ \frac{\pi^2(m^2 + n^2)}{L^2} : (m, n) \in \mathbb{N}^2, 1 \leq m < n \right\}, \\ \mathcal{N}_2 &:= \left\{ \frac{\pi^2 m^2}{L^2} : m \in \mathbb{N}, 1 \leq m \right\}, \\ \mathcal{N}_3 &:= \left\{ \frac{\pi^2}{L^2} \left(n + \frac{1}{2} \right)^2 : n \in \mathbb{N}, n \geq 0 \right\}, \\ \mathcal{N}_4 &:= \left\{ \frac{\pi^2}{L^2} \left(m^2 + \left(n + \frac{1}{2} \right)^2 \right) : (m, n) \in \mathbb{N}^2, 1 \leq m, 0 \leq n \right\}, \\ \mathcal{N}_{odd} &:= \left\{ \frac{\pi^2}{4L^2} ((2m+1)^2 + (2n+1)^2) : (m, n) \in \mathbb{N}^2, 0 \leq m < n \right\}. \end{aligned}$$

To simplify the presentation of our main results let us also introduce the notations

$$\mathcal{N}_{even} := \mathcal{N}_1 \cup \mathcal{N}_2,$$

$$\mathcal{N}_{mixt} := \mathcal{N}_3 \cup \mathcal{N}_4.$$

Observe that

$$(6) \quad \mathcal{N}_{even} \cup \mathcal{N}_{odd} = \frac{\mathcal{N}_0}{4L^2}.$$

A priori, for the models we study here each control possesses N components but some of the inputs might not be necessary and could vanish completely. In fact, in the case when the system is null-controllable the goal is to intend the controls to act on a minimal number of components. Our main results will be stated accurately in the following. Roughly speaking, for problem (2)-(I) we prove that the null controllability property holds with $N - 1$ control inputs, whereas for the model (2)-(II) we need $2N - 1$ control inputs. In both cases the control properties are obtained under some restrictions on the parameter λ . For our purposes we need to work in a rigorous functional framework in which Sobolev spaces play a crucial

role, namely $L^2(\Gamma)$ and $H^m(\Gamma)$ for $m \in \mathbb{N}^*$. By $L^2(\Gamma)$ and $H^m(\Gamma)$ we understand the Hilbert spaces

$$L^2(\Gamma) := \prod_{i=1}^N L^2(0, L), \quad H^m(\Gamma) := \prod_{i=1}^N H^m(0, L), \quad m \geq 1,$$

endowed with their natural norms.

Thus, our main results are as follows.

Theorem 1.1 (Null-controllability for model (2)-(I)). *Let $T > 0$ be fixed. For any $\lambda \notin \mathcal{N}_{\text{even}} \cup \mathcal{N}_{\text{odd}}$ and any initial state $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$ there exists a control $u = (u^k)_{k=1,N} \in H^1(\Gamma)$ having at most $N-1$ non-identically vanishing components such that the solution of system (2)-(I) satisfies*

$$(7) \quad y^k(T, x) = 0, \text{ for any } x \in (0, L), \quad k \in \{1, \dots, N\}.$$

Theorem 1.2 (Null-controllability for model (2)-(II)). *Let $T > 0$ be fixed. For any $\lambda \notin \mathcal{N}_{\text{even}} \cup \mathcal{N}_{\text{mixt}}$ and any initial state $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$ there exist the controls $a = (a^k)_{k=1,N}, b = (b^k)_{k=1,N} \in H^1(\Gamma)$ having at most $2N-1$ non-identically vanishing components such that the solution of system (2)-(II) satisfies*

$$(8) \quad y^k(T, x) = 0, \text{ for any } x \in (0, L), \quad k \in \{1, \dots, N\}.$$

In order to prove Theorems 1.1 and 1.2 we make a careful spectral analysis of the corresponding elliptic differential operator which allows us to transform the controllability problem into an equivalent moment problem. The later will be solved combining a general moment theory developed by Fattorini and Russell [12] and the asymptotic analysis of the eigenvalues. At that point it is important to note that, for making possible the existence of the controls, the choice of the parameter λ plays an important role. It is also worth mentioning that the general theory in [12] allows to build solutions to the moment problem in L^2 . However, as the authors in [12] assert later on one can obtain the existence of smoother controls not only in L^2 but in any space H^s , $s \in \mathbb{R}$ and in consequence in $C^\infty([0, T])$.

The extension of the above results to the case when the edges of the star shaped tree have different lengths needs a different approach. Following [11] other coupling conditions may be imposed at the internal node. The analysis of the controllability properties for the KS system with other coupling conditions at $x = 0$ remains to be considered elsewhere.

The analysis described above is organized in two sections: in Section 2 we study problem (2)-(I) and prove Theorem 1.1. Section 3 is devoted to problem (2)-(II) and the proof of Theorem 1.2.

2. THE KS EQUATION OF TYPE (I)

The controllability problem (2)-(I) will be studied by using the method of moments due to Fattorini and Russell [12]. Therefore, a careful spectral analysis of the involved elliptic operator is necessary.

2.1. Spectral analysis. For any $\lambda > 0$ let us consider the following spectral problem on Γ :

$$(9) \quad \begin{cases} \lambda \phi_{xx}^k + \phi_{xxxx}^k = \sigma \phi^k, & x \in (0, L) \\ \phi^i(0) = \phi^j(0), & i, j \in \{1, \dots, N\} \\ \phi^k(L) = \phi_x^k(L) = 0, & k \in \{1, \dots, N\} \\ \sum_{k=1}^N \phi_x^k(0) = 0, \\ \phi_{xx}^i(0) = \phi_{xx}^j(0), & i, j \in \{1, \dots, N\} \\ \sum_{k=1}^N \phi_{xxx}^k(0) = 0. \end{cases}$$

To begin with, we introduce the forth order operator

$$A : D(A) \subset L^2(\Gamma) \rightarrow L^2(\Gamma)$$

given by

$$(10) \quad \begin{cases} A\phi^k = \lambda \phi_{xx}^k + \phi_{xxxx}^k, & k \in \{1, \dots, N\} \\ D(A) = \left\{ \begin{array}{l} \phi = (\phi^k)_{k=1, N} \in H^4(\Gamma) \mid \phi^k(L) = \phi_x^k(L) = 0, \quad k \in \{1, \dots, N\}, \\ \phi^i(0) = \phi^j(0), \quad \phi_{xx}^i(0) = \phi_{xx}^j(0), \quad i, j \in \{1, \dots, N\}, \\ \sum_{k=1}^N \phi_x^k(0) = 0, \quad \sum_{k=1}^N \phi_{xxx}^k(0) = 0 \end{array} \right\}. \end{cases}$$

Remark that spectral problem (9) is equivalent to

$$(11) \quad \begin{cases} A\phi = \sigma \phi, \\ \phi \in D(A). \end{cases}$$

To study this eigenvalue problem we firstly claim that

Proposition 2.1. *For any $\mu > \lambda^2/4$ the operator*

$$A + \mu I : D(A) \subset L^2(\Gamma) \rightarrow L^2(\Gamma),$$

is a non-negative self-adjoint operator with compact inverse. In particular, it has a pure discrete spectrum consisted by a sequence of nonnegative eigenvalues $\{\sigma_{\mu, l}\}_{l \in \mathbb{N}}$ satisfying $\lim_{l \rightarrow \infty} \sigma_{\mu, l} = \infty$. Moreover, up to a normalization, the corresponding eigenfunctions $\{\phi_{\mu, l}\}_{l \in \mathbb{N}}$ form an orthonormal basis of $L^2(\Gamma)$.

Proof. Firstly, it is easy to observe that $D(A)$ is dense in $L^2(\Gamma)$. Next the proof will be done in several steps.

Step 1: A is a symmetric operator. Indeed, after integrations by parts, we get

$$\begin{aligned} (Au, v)_{L^2(\Gamma)} &= (u, Av)_{L^2(\Gamma)} \\ &= \sum_{k=1}^N \int_0^L (u_{xx}^k v_{xx}^k - \lambda u_x^k v_x^k) dx, \quad \forall u, v \in D(A). \end{aligned}$$

Step 2: $A + \mu I$ is maximal monotone for any $\mu > \lambda^2/4$. Firstly, let us show the monotonicity property

$$((A + \mu I)u, u)_{L^2(\Gamma)} > 0, \quad \forall u \in D(A).$$

Indeed,

$$(12) \quad ((A + \mu I)u, u)_{L^2(\Gamma)} = \sum_{k=1}^N \int_0^L (|u_{xx}^k|^2 + \mu |u^k|^2 - \lambda |u_x^k|^2) dx.$$

On the other hand we have

$$\begin{aligned} \sum_{k=1}^N \int_0^L u_{xx}^k u^k dx &= \sum_{k=1}^N u_x^k u^k \Big|_{x=0}^{x=L} - \sum_{k=1}^N \int_0^L |u_x^k|^2 dx \\ &= - \sum_{k=1}^N \int_0^L |u_x^k|^2 dx. \end{aligned}$$

Therefore, from the inequality of arithmetic and geometric means it holds

$$(13) \quad \sum_{k=1}^N \int_0^L |u_x^k|^2 dx \leq \sum_{k=1}^N \int_0^L \left(\frac{1}{\lambda} |u_{xx}^k|^2 + \frac{\lambda}{4} |u^k|^2 \right) dx.$$

Combining (12) and (13) we obtain

$$((A + \mu I)u, u)_{L^2(\Gamma)} \geq \left(\mu - \frac{\lambda^2}{4} \right) \sum_{k=1}^N \int_0^L |u^k|^2 dx,$$

which is positive provided $\mu > \lambda^2/4$.

Next we emphasize that $A + \mu I$ is maximal, i.e., for any $f \in L^2(\Gamma)$ there exists a unique $u \in D(A)$ such that $(A + \mu I)u = f$. To do that, first we consider the variational formulation

$$(14) \quad \begin{cases} a(u, v) = (f, v)_{L^2(\Gamma)}, & \forall v \in V \\ u \in V, \end{cases}$$

where V denotes the Hilbert space

$$V = \left\{ \begin{array}{l} \phi = (\phi^k)_{k=1, N} \in H^2(\Gamma) \mid \phi^k(L) = \phi_x^k(L) = 0, \quad k \in \{1, \dots, N\}, \\ \phi^i(0) = \phi^j(0), \quad i, j \in \{1, \dots, N\}, \quad \sum_{k=1}^N \phi_x^k(0) = 0 \end{array} \right\}$$

endowed with the $H^2(\Gamma)$ -norm and $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ denotes the bilinear form

$$a(u, v) = \sum_{k=1}^N \int_0^L (u_{xx}^k v_{xx}^k - \lambda u_x^k v_x^k + \mu u^k v^k) dx.$$

It is easy to see that $a(\cdot, \cdot)$ is symmetric and continuous. In addition, it is also coercive. Indeed, let $\delta > 0$ small enough such that $\mu > \lambda^2/(4 - 4\delta)$. Then, arguing as in (13) we obtain

$$\sum_{k=1}^N \int_0^L |u_x^k|^2 dx \leq \sum_{k=1}^N \int_0^L \left(\frac{1-\delta}{\lambda} |u_{xx}^k|^2 + \frac{\lambda}{4(1-\delta)} |u^k|^2 \right) dx,$$

which together with (12) leads to

$$a(u, u) \geq \delta \sum_{k=1}^N \int_0^L |u_{xx}^k|^2 dx + \left(\mu - \frac{\lambda^2}{4 - 4\delta} \right) \sum_{k=1}^N \int_0^L |u^k|^2 dx,$$

and so a is coercive. Applying Lax-Milgram lemma we ensure the existence of a unique $u \in V$ satisfying the variational problem (14). In order to justify the maximality of $A + \mu I$ it is sufficient to show that the solution u of (14) belongs actually to $D(A)$. For that, we refer to the classical regularity arguments for elliptic operators (see, for instance, [1]).

Step 3: A is a self-adjoint operator with compact inverse. Since $A + \mu I : D(A) \subset L^2(\Gamma) \rightarrow L^2(\Gamma)$ is a symmetric operator and maximal monotone, it is a self-adjoint operator (see, e.g., [1]).

Moreover, we have that the linear operator $(A + \mu I)^{-1} : L^2(\Gamma) \rightarrow L^2(\Gamma)$, given by

$$(A + \mu I)^{-1} f = u \in D(A) \subset L^2(\Gamma),$$

satisfies

$$\|(A + \mu I)^{-1} f\|_{H^2(\Gamma)} \leq C \|f\|_{L^2(\Gamma)},$$

for some positive constant $C > 0$. Since the embedding $H^2(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact it follows that $(A + \mu I)^{-1}$ is a compact operator. Then applying the classical spectral results for compact self-adjoint operators we conclude the proof of Proposition 2.1. \square

Remark 2.2. The spectrum $\{\sigma_l\}_{l \in \mathbb{N}}$ of problem (11) is obtained by shifting the spectrum of $A + \mu I$ in Proposition 2.1, i.e.,

$$\sigma_l := \sigma_{\mu, l} - \mu, \quad \forall l \in \mathbb{N}.$$

In particular, it holds that

$$(15) \quad -\frac{\lambda^2}{4} \leq \sigma_l, \quad \forall l \in \mathbb{N}, \quad \sigma_l \rightarrow \infty, \text{ as } l \rightarrow \infty.$$

The following proposition will be also very useful in our analysis.

Proposition 2.3. The lower bound $-\lambda^2/4$ is not an eigenvalue for the operator A .

Proof. If $(-\lambda^2/4, \phi)$ were an eigenpair for A , then $A\phi = \lambda^2/4\phi$ for some nontrivial $\phi = (\phi^k)_{k=1, N} \in D(A)$. Then, integration by parts leads to

$$0 = \left(\left(A + \frac{\lambda^2}{4} \right) \phi, \phi \right)_{L^2(\Gamma)} = \sum_{k=1}^N \int_0^L \left(\phi_{xx}^k + \frac{\lambda}{2} \phi^k \right)^2 dx.$$

Therefore, for any $k \in \{1, \dots, N\}$ the component ϕ^k must satisfy the equation $\phi_{xx}^k + \frac{\lambda}{2} \phi^k = 0$ in $(0, L)$ subject to the boundary conditions $\phi^k(L) = \phi_x^k(L) = 0$. These allow us to obtain $\phi^k \equiv 0$, for all $k \in \{1, \dots, N\}$ which is in contradiction with $\phi \neq 0$. \square

2.2. Qualitative properties of the eigenvalues. The main goal of this section is to provide an asymptotic formula for the behavior of eigenvalues of system (9). Particularly, this result will play an important role to prove the null-controllability of problem (2)-(I).

For any fixed $\lambda > 0$ let us firstly consider the following two eigenvalue problems on the interval $(0, L)$:

$$(16) \quad \begin{cases} \lambda \Psi_{xx} + \Psi_{xxxx} = \sigma \Psi, & x \in (0, L), \\ \Psi_x(0) = 0, \Psi_{xxx}(0) = 0, \\ \Psi(L) = \Psi_x(L) = 0 \end{cases}$$

and

$$(17) \quad \begin{cases} \lambda \Phi_{xx} + \Phi_{xxxx} = \sigma \Phi, & x \in (0, L), \\ \Phi(0) = 0, \Phi_{xx}(0) = 0, \\ \Phi(L) = \Phi_x(L) = 0, \end{cases}$$

respectively. As in Section 2.1 we can easily show that systems (16)-(17) possess a sequence of eigenvalues $\{\sigma_n\}_n$ which tends to infinity and is strictly bounded from

below by $-\lambda^2/4$. Before going through let us fix some notations which will be useful in the forthcoming sections. For any $\lambda > 0$ we denote

$$(18) \quad \begin{cases} \alpha := \sqrt{\frac{-\lambda + \sqrt{\lambda^2 + 4\sigma}}{2}}, & \beta := \sqrt{\frac{\lambda + \sqrt{\lambda^2 + 4\sigma}}{2}}, & \text{if } \sigma \geq 0 \\ \gamma := \sqrt{\frac{\lambda - \sqrt{\lambda^2 + 4\sigma}}{2}}, & \beta := \sqrt{\frac{\lambda + \sqrt{\lambda^2 + 4\sigma}}{2}}, & \text{if } -\frac{\lambda^2}{4} < \sigma < 0 \end{cases}$$

for which we have the relations

$$(19) \quad \begin{cases} \beta^2 - \alpha^2 = \lambda, & \text{if } \sigma \geq 0 \\ \beta^2 + \gamma^2 = \lambda, & \text{if } -\frac{\lambda^2}{4} < \sigma < 0 \end{cases}$$

and

$$(20) \quad \sigma = \begin{cases} \alpha^2 \beta^2, & \text{if } \sigma \geq 0 \\ -\beta^2 \gamma^2, & \text{if } -\frac{\lambda^2}{4} < \sigma \leq 0. \end{cases}$$

Coming back to our spectral problem (9), we introduce the functions

$$(21) \quad S := \sum_{k=1}^N \phi^k$$

and

$$(22) \quad D^k := \phi^k - \frac{S}{N}, \quad k \in \{1, \dots, N\}.$$

The motivation for analyzing systems (16) and (17) is due to the fact that S verifies (16) whereas D^k satisfies (17) for all $k \in \{1, \dots, N\}$.

Next we state and prove some preliminary results.

Lemma 2.4. *For any $\lambda > 0$ the eigenvalue problems (16) and (17) have no any common eigenvalue σ .*

Proof. Let us assume that there exists σ and two functions Ψ and Φ not identically vanishing, satisfying (16) and (17), respectively.

The boundary conditions at $x = 0$ in (16) and (17) allow to introduce the even and odd extensions of Ψ , respectively Φ , with respect to $x = 0$. More precisely, we consider

$$\overline{\Psi}(x) := \begin{cases} \Psi(x), & x \in [0, L] \\ \Psi(-x), & x \in [-L, 0] \end{cases}$$

and

$$\overline{\Phi}(x) := \begin{cases} \Phi(x), & x \in [0, L] \\ -\Phi(-x), & x \in [-L, 0]. \end{cases}$$

Then $\overline{\Psi}$ and $\overline{\Phi}$ verify

$$(23) \quad \begin{cases} \lambda \overline{\Psi}_{xx} + \overline{\Psi}_{xxxx} = \sigma \overline{\Psi}, & x \in (-L, L) \\ \overline{\Psi}(-L) = \overline{\Psi}(L) = \overline{\Psi}_x(-L) = \overline{\Psi}_x(L) = 0 \end{cases}$$

and

$$(24) \quad \begin{cases} \lambda \overline{\Phi}_{xx} + \overline{\Phi}_{xxxx} = \sigma \overline{\Phi}, & x \in (-L, L) \\ \overline{\Phi}(-L) = \overline{\Phi}(L) = \overline{\Phi}_x(-L) = \overline{\Phi}_x(L) = 0, \end{cases}$$

respectively. Finally, let us denote

$$\hat{\Psi}(y) := \overline{\Psi}(2Ly - L), \quad \hat{\Phi}(y) := \overline{\Phi}(2Ly - L), \quad y \in (0, 1).$$

In view of (23) and (24) it follows that $\hat{\Psi}$ and $\hat{\Phi}$ satisfy the same eigenvalue problem

$$(25) \quad \begin{cases} \lambda(2L)^2 \phi_{xx} + \phi_{yyyy} = \sigma(2L)^4 \phi, & y \in (0, 1) \\ \phi(0) = \phi(1) = \phi_y(0) = \phi_y(1) = 0. \end{cases}$$

The arguments in [4] show that problem (25) admits simple eigenvalues. This means that $\hat{\Phi} = \alpha \hat{\Psi}$ for some constant $\alpha \neq 0$ and, equivalently, we have $\bar{\Phi} = \alpha \bar{\Psi}$. Since $\bar{\Psi}$ is an even function and $\bar{\Phi}$ is an odd function (both of them vanishing on the boundary) we necessarily have $\bar{\Psi} = \bar{\Phi} \equiv 0$, which contradicts our assumptions. \square

In consequence we have the following partition for the eigenvalues of system (9).

Lemma 2.5. *For any given $\lambda > 0$, σ is an eigenvalue for system (9) if and only if σ is an eigenvalue for either (16) or (17).*

Proof. “Only if” implication. Assume that $(\sigma, \phi = (\phi^k)_{k=1,N})$ is an eigenpair of (9). Then σ verifies (16) for $\Psi = S$ in (21). In addition, σ verifies (17) for any $\Phi = D^k$ in (22). If $S \neq 0$ then (σ, S) is an eigenpair for (16). Otherwise, if $S = 0$ then $D^k = \phi^k$ for all $k \in \{1, \dots, N\}$. From the initial assumption there exists $k_0 \in \{1, \dots, N\}$ such that $\phi^{k_0} \neq 0$, and therefore, (σ, ϕ^{k_0}) is an eigenpair for (17).

“If” implication. Let (σ, Ψ) be an eigenpair of (16). Then $(\sigma, \phi = (\Psi, \dots, \Psi, \Psi))$ is an eigenpair of (9). Let (σ, Φ) be an eigenpair of (17). Then it follows that $(\sigma, \phi = (0, \dots, 0, -\Phi, \Phi))$ is an eigenpair of (9), which completes the proof of Lemma 2.5. \square

Lemma 2.6. *For any $\lambda(2L)^2 \notin \mathcal{N}_0$ any eigenfunction $\phi = (\phi^k)_{k=1,N}$ of A satisfies $\phi_{xx}^k(L) \neq 0$ for at least two indexes $k \in \{1, \dots, N\}$.*

Proof. With the same notations as above we have that S and D^k , $k \in \{1, \dots, N\}$, satisfy (16) and (17). We distinguish two cases as follows.

The case $S \neq 0$. Using Lemma 2.4 we must have $D^k \equiv 0$, for all $k \in \{1, \dots, N\}$. This means that $\phi = 1/N(S, \dots, S)$ where S is the eigenfunction of problem (16). Using [4, Lemma 2.1] we have that under the assumption $\lambda(2L)^2 \notin \mathcal{N}_0$ the eigenfunction \hat{S} of (25) satisfies $\hat{S}_{xx}(0) \neq 0$. Due to the symmetry of the boundary conditions we notice that the function $x \mapsto \hat{S}(1-x)$ is also an eigenfunction of problem (25). Since any eigenvalue is simple we deduce that $\hat{S}(x) = \hat{S}(1-x)$ and therefore we also have $\hat{S}_{xx}(1) \neq 0$. This gives us the desired property for S , $S_{xx}(L) \neq 0$ and the problem is solved.

The case $S \equiv 0$. In this case we have $D^k = \phi^k$ for all $k \in \{1, \dots, N\}$. Assume that for at least $N-1$ indexes $k \in \{1, \dots, N\}$ we have $\phi_{xx}^k(L) = 0$. Then we must have $D_{xx}^k(L) = \phi_{xx}^k(L) = 0$ for all $k \in \{1, \dots, N\}$. On the other hand, we know that there exists $k_0 \in \{1, \dots, N\}$ satisfying $\phi^{k_0} \neq 0$, which implies that D^{k_0} is an eigenfunction for (17). Since $\lambda(2L)^2 \notin \mathcal{N}_0$, by using the same argument from [4, Lemma 2.1] as above we must have $D_{xx}^{k_0}(L) \neq 0$, which leads to a contradiction.

Therefore, the proof is finished. \square

Lemma 2.7. *The eigenvalues of problem (9) satisfy*

$$\sigma_n = \left(\frac{\pi}{NL}\right)^4 n^4 + O(n^3), \quad n \rightarrow \infty.$$

Proof. Since there exists a finite number of non-positive eigenvalues we just concentrate on the positive eigenvalues σ .

Let us consider $(\sigma, \phi = (\phi^1, \dots, \phi^N))$ an eigenpair of (9) with $\sigma > 0$. Then S satisfies (16) and each D^k satisfies (17). Since the eigenvalue problems (16) and (17) have no common eigenvalues according to Lemma 2.4 we distinguish two cases: $S \equiv 0$ or $S \neq 0$ and $D^k \equiv 0$ for all $k \in \{1, \dots, N\}$.

In the second case we have $\phi = (\phi^1, \dots, \phi^1)$ where ϕ^1 is an eigenfunction for problem (16). Since $\sigma > 0$, with the notations in (18) then

$$\phi^1(x) = C_1 \cos(\beta x) + C_2 \sin(\beta x) + C_3 \cosh(\alpha x) + C_4 \sinh(\alpha x),$$

where C_1, C_2, C_3, C_4 are such that to satisfy the boundary conditions in (16) and $\phi^1 \not\equiv 0$. From the conditions at $x = 0$ we get $C_2 = C_4 = 0$. From the conditions at $x = L$ we obtain $\phi^1 \not\equiv 0$ when the compatibility conditions

$$(26) \quad \beta \cosh(\alpha L) \sin(\beta L) + \alpha \sinh(\alpha L) \cos(\beta L) = 0, \quad \alpha, \beta > 0,$$

are satisfied. This is equivalent to

$$\beta \tan(\beta L) = -\alpha \tanh(\alpha L), \quad \cos(\beta L) \neq 0$$

or equivalently (in view of (18))

$$\sqrt{\lambda + \alpha^2} \tan(\sqrt{\lambda + \alpha^2} L) = -\alpha \tanh(\alpha L), \quad \cos(\sqrt{\lambda + \alpha^2} L) \neq 0.$$

It is not difficult to note that the function $(0, \infty) \ni \alpha \mapsto -\alpha \tanh(\alpha L)$ is strictly decreasing whereas the function $(0, \infty) \setminus \{\alpha \mid \sqrt{\lambda + \alpha^2} L = n\pi + \pi/2, n \in \mathbb{N}\} \ni \alpha \mapsto \sqrt{\lambda + \alpha^2} \tan(\sqrt{\lambda + \alpha^2} L)$ is increasing on each interval of the domain (for that to be proved we just have to look at the sign of the corresponding derivatives, see also [10] for similar arguments). So, there exists two sequences $\{\alpha_{1,n}\}_{n \geq 0}$ and $\{\beta_{1,n}\}_{n \geq 0}$ with $\beta_{1,n}^2 = \lambda + \alpha_{1,n}^2$, solutions for (26), where $\beta_{1,n} L \in (n\pi - \pi/2, n\pi + \pi/2)$. Then the sequence of positive eigenvalues $\{\sigma_{1,n}\}_{n \geq 0}$ is given by

$$(27) \quad \sigma_{1,n} = \beta_{1,n}^2 \alpha_{1,n}^2.$$

and therefore $\sigma_{1,n} = (\frac{\pi}{NL})^4 n^4 + O(n^3)$. Moreover, by using the arguments in [4] we have that these eigenvalues are simple.

Let us now consider the case $S \equiv 0$. In view of Lemma 2.5 any ϕ^k is an eigenfunction of (17). Since the eigenvalues of (17) are simple there exist some constants c_k such that $\phi^k = c_k \phi^1$ for any $k \in \{1, \dots, N\}$. The requirement for σ to be an eigenvalue of (17) is equivalent to

$$\beta \sinh(\alpha L) \cos(\beta L) - \alpha \cosh(\alpha L) \sin(\beta L) = 0$$

or equivalently

$$(28) \quad \frac{1}{\alpha} \tanh(\alpha L) = \frac{1}{\beta} \tan(\beta L), \quad \text{with } \beta = \sqrt{\lambda + \alpha^2}, \quad \cos(\beta L) \neq 0.$$

In the same way as in the previous case it is not difficult to prove that the function $(0, \infty) \ni \alpha \mapsto \tanh(\alpha L)/\alpha$ is decreasing whereas the function $(0, \infty) \setminus \{\alpha \mid \sqrt{\lambda + \alpha^2} L = n\pi + \pi/2, n \in \mathbb{N}\} \ni \alpha \mapsto \tan(\sqrt{\lambda + \alpha^2} L)/(\sqrt{\lambda + \alpha^2} L)$ is strictly increasing on each open interval of the domain. Thus we obtain two sequences of solutions $\{\alpha_{2,n}\}_{n \geq 0}$ $\{\beta_{2,n}\}_{n \geq 0}$, to (28) with $\beta_{2,n} L \in (n\pi - \pi/2, n\pi + \pi/2)$. Then we get a set of corresponding eigenvalues which satisfy $\sigma_{2,n} = (\frac{\pi}{NL})^4 n^4 + O(n^3)$. Moreover, since $S \equiv 0$ we must have $\sum_{k=1}^N c_k = 0$. This shows that the eigenvalues $\sigma_{2,n}$ have multiplicity $N - 1$. Counting the number of eigenvalues in the interval $[0, r]$, $r > 0$, we obtain finally the desired result. \square

2.3. Well-posedness. In order to study the well-posedness of problem (2)-(I) we apply the semigroup theory. First, let us consider the polynomial

$$(29) \quad P(x) = \left(\frac{x}{L}\right)^4 (x - L)$$

and denote $z^k = y^k - P(x)u^k(t)$, $k \in \{1, \dots, N\}$. For our purpose it is more convenient to analyze first the equation satisfied by $z = (z^k)_{k=1, N}$. Indeed, it

is easy to see that z satisfies the following nonhomogeneous problem with zero boundary conditions:

$$(30) \quad \begin{cases} z_t^k + \lambda z_{xx}^k + z_{xxx}^k = -P u_t^k(t) - (\lambda P_{xx} + P_{xxx}) u^k(t), & (t, x) \in (0, T) \times (0, L) \\ z^i(t, 0) = z^j(t, 0), & t \in (0, T), \quad i, j \in \{1, \dots, N\} \\ z^k(t, L) = z_x^k(t, L) = 0, & k \in \{1, \dots, N\} \\ \sum_{k=1}^N z_x^k(t, 0) = 0, & t \in (0, T) \\ z_{xx}^i(t, 0) = z_{xx}^j(t, 0), & t \in (0, T), \quad i, j \in \{1, \dots, N\} \\ \sum_{k=1}^N y_{xxx}^k(t, 0) = 0, & t \in (0, T) \\ z^k(0, x) = y_0^k(x) - P(x) u^k(0), & x \in (0, L). \end{cases}$$

Problem (30) can be written as an abstract Cauchy problem. Indeed, it follows that

$$(31) \quad \begin{cases} z_t + Az = F(t, x, u), & t \in (0, T) \\ z(0) = z_0, \end{cases}$$

where A is the operator defined in (10), whereas $F = (F^k)_{k=1, N}$ and $z_0 = (z_0^k)_{k=1, N}$ are given by

$$F^k(t, x, u) = -P(x) u_t^k(t) - (\lambda P_{xx}(x) + P_{xxx}(x)) u^k(t),$$

respectively,

$$z_0^k(x) = y_0^k(x) - P(x) u^k(0),$$

for any $k \in \{1, \dots, N\}$. In the previous section we have proved that $(A, D(A))$ generates a semigroup in $L^2(\Gamma)$. Therefore, applying the Hille-Yosida theory for the Cauchy problem (31) (see, e.g., [3, Proposition 4.1.6 and Lemma 4.1.5]) we finally obtain

Proposition 2.8. *If $z_0 \in D(A)$ and $F \in C([0, T], L^2(\Gamma)) \cap L^1((0, T), D(A))$ there exists a function $z \in C([0, T], D(A)) \cap C^1([0, T], L^2(\Gamma))$ solution to (30). Moreover, if $z_0 \in L^2(\Gamma)$ and $F \in C([0, T], L^2(\Gamma))$ (it can be extended to $L^1((0, T), L^2(\Gamma))$) there exists a mild function $z \in C([0, T], L^2(\Gamma))$ solution to (30).*

Proposition 2.8 extends for y solution of (2) with initial data $y_0 \in D(A)$ and $y_0 \in L^2(\Gamma)$, respectively.

2.4. Controllability problem. Next we address the controllability problem (2)-(I) by using the method of moments [12]. In view of that, let us consider the

so-called adjoint problem, that is

$$(32) \quad \begin{cases} -q_t^k + \lambda q_{xx}^k + q_{xxx}^k = 0, & (t, x) \in (0, T) \times (0, L), \\ q^i(t, 0) = q^j(t, 0), & t \in (0, T), \quad i, j \in \{1, \dots, N\}, \\ q^k(t, L) = q_x^k(t, L) = 0, & t \in (0, T), \quad k \in \{1, \dots, N\}, \\ \sum_{k=1}^N q_x^k(t, 0) = 0, & t \in (0, T), \\ q_{xx}^i(t, 0) = q_{xx}^j(t, 0), & t \in (0, T), \quad i, j \in \{1, \dots, N\}, \\ \sum_{k=1}^N q_{xxx}^k(t, 0) = 0, & t \in (0, T), \\ q^k(T, x) = q_T^k(x), & x \in (0, L), \end{cases}$$

Then, we have the following characterization of the null-controllability property.

Lemma 2.9. *The system (2) is null-controllable in time $T > 0$ if and only if, for any initial data $y_0 = (y_0^k)_{k=1, N} \in L^2(\Gamma)$ there exists a control function $u = (u^k)_{k=1, N} \in H^1(\Gamma)$ such that, for any $q_T = (q_T^k)_{k=1, N} \in L^2(\Gamma)$*

$$(33) \quad (y_0, q(0))_{L^2(\Gamma)} = \sum_{k=1}^N \int_0^T u^k(t) q_{xx}^k(t, L) dt,$$

where q is the solution of (32).

Proof. We proceed as in the classical duality approach. We first multiply the equation in (2) by $q = (q^k)_{k=1, N}$, the solution of (32) to obtain

$$\sum_{k=1}^N \int_0^T \int_0^L (y_t^k + \lambda y_{xx}^k + y_{xxx}^k) q^k dx dt = 0.$$

Integration by parts leads to

$$(34) \quad \begin{aligned} 0 = & \sum_{k=1}^N \int_0^T y^k q^k \Big|_{t=0}^{t=T} dx + \sum_{k=1}^N \int_0^T \int_0^L (-q_t^k + \lambda q_{xx}^k + q_{xxx}^k) y^k dx dt \\ & + \int_0^T \left(\lambda y_x^k q^k \Big|_{x=0}^{x=L} - \lambda y^k q_x^k \Big|_{x=0}^{x=L} + y_{xxx}^k q^k \Big|_{x=0}^{x=L} \right. \\ & \left. - y_{xx}^k q_x^k \Big|_{x=0}^{x=L} + y_x^k q_{xx}^k \Big|_{x=0}^{x=L} - y^k q_{xxx}^k \Big|_{x=0}^{x=L} \right) dx dt. \end{aligned}$$

In view of the boundary conditions satisfied by $y = (y^k)_{k=1, N}$ and $q = (q^k)_{k=1, N}$, identity (34) is equivalent to

$$(35) \quad \sum_{i=1}^N \int_0^L (y^k(T, x) q^k(T, x) - y_0^k(x) q_0^k(x)) dx + \int_0^T u^k(t) q_{xx}^k(t, L) dt = 0.$$

“Only if” implication. Since (2) is null-controllable (i.e. $y(T, x) = 0$ for any $x \in \Gamma$) it follows from (35) that condition (33) holds true.

“If” implication. Let us assume the validity of (33). In this case, due to (35) we get

$$(y(T), q_T)_{L^2(\Gamma)} = 0,$$

for any $q_T \in L^2(\Gamma)$. This implies $y(T) = 0$. \square

As a consequence of Lemma 2.9 and the spectral analysis developed in subsection 2.1 the controllability problem reduces to the following moment problem.

Lemma 2.10. *Let $\{\phi_l\}_{l \in \mathbb{N}}$ be the orthonormal basis of $L^2(\Gamma)$ formed by the eigenfunctions of A corresponding to the eigenvalues $\{\sigma_l\}_{l \in \mathbb{N}}$. Then system (2) is null-controllable if for any initial data $y_0 \in L^2(\Gamma)$,*

$$(36) \quad y_0 = \sum_{l \in \mathbb{N}} y_{0,l} \phi_l,$$

and any time $T > 0$, there exists a control $u = (u^k)_{k=1,N} \in H^1(\Gamma)$ such that

$$(37) \quad y_{0,l} e^{-T\sigma_l} = \sum_{k=1}^N \phi_{l,xx}^k(L) \int_0^T u^k(T-t) e^{-t\sigma_l} dt, \quad \forall l \in \mathbb{N}.$$

Proof. For any $q_T \in L^2(\Gamma)$ we have

$$q_T = \sum_{l \in \mathbb{N}} q_l \phi_l,$$

where $\sum_{l \in \mathbb{N}} |q_l|^2 < \infty$. Then, seeking for solutions in separable variable

$$q(t, x) = \sum_{l \in \mathbb{N}} \bar{q}_l(t) \phi_l(x),$$

the time coefficients \bar{q}_l satisfy

$$\bar{q}_l' - \sigma_l \bar{q}_l = 0, \quad \bar{q}_l(T) = q_l.$$

Then, we obtain

$$q(t, x) = \sum_{l \in \mathbb{N}} e^{(-T+t)\sigma_l} q_l \phi_l(x),$$

and therefore

$$(38) \quad q_{xx}(t, L) = \sum_{l \in \mathbb{N}} e^{(-T+t)\sigma_l} q_l \phi_{l,xx}(L).$$

Plugging (38) and (36) in the controllability condition (33) we obtain that the existence of a function u satisfying the moment problem (37) suffices to prove the null-controllability property. \square

2.5. Proof of Theorem 1.1. We show that a control acting on $N - 1$ nodes is sufficient to obtain the null-controllability of system (2)-(I).

According to Lemma 2.10 we have to solve the moment problem (37) by constructing a control u . In order to do that we settle one of the components of $u = (u^k)_{k=1,N}$ to be identically zero. For simplicity, we assume $u^N = 0$. Then the problem of moments (37) becomes

$$y_{0,l} e^{-T\sigma_l} = \sum_{k=1}^{N-1} \phi_{l,xx}^k(L) \int_0^T u^k(T-t) e^{-t\sigma_l} dt, \quad \forall l \in \mathbb{N}.$$

Let $M(l) = \{k \in \{1, \dots, N-1\} \mid \phi_{l,xx}^k(L) \neq 0\}$. Due to Lemma 2.6 we have $M(l) \neq \emptyset$ for any $l \in \mathbb{N}$. Then the control inputs $u^k \in L^2(0, T)$, $k \in \{1, \dots, N-1\}$ should satisfy the following moment problems

$$(39) \quad \begin{cases} \int_0^T u^k(T-t) e^{-T\sigma_l} dt = \frac{y_{0,l} e^{-T\sigma_l}}{|M(l)| \phi_{l,xx}^k(L)}, & \text{if } k \in M(l), \quad \forall l \\ \int_0^T u^k(T-t) e^{-T\sigma_l} dt = 0, & \text{if } k \notin M(l), \quad \forall l. \end{cases}$$

Thanks to the asymptotic behavior of the eigenvalues given by $\{\sigma_l\}_l$ in Lemma 2.7 we can apply the moment method employed in [12, Corollary 3.2] to construct the controls $(u^k)_{k=1,N-1}$ verifying (39) and finalize the proof.

3. THE KS EQUATION OF TYPE (II)

The results obtained in this section are based on a careful analysis of the eigenvalues for the corresponding elliptic operator of system (2)-(II). In the previous section we addressed this problem for system (2)-(I) by using specific spectral results done by Cerpa [4]. In the present case such results are not applicable. Therefore, to take advantage of the strategy implemented in the previous case an additional work has to be done by determining explicitly the spectrum and the eigenfunctions for two different eigenvalue problems as follows.

3.1. Preliminaries I. In this subsection we analyze the following eigenvalue problem

$$(40) \quad \begin{cases} \lambda \phi_{xx} + \phi_{xxxx} = \sigma \phi, & x \in (0, L) \\ \phi_x(0) = \phi_{xxx}(0) = 0, \\ \phi_x(L) = \phi_{xxx}(L) = 0. \end{cases}$$

Again, we can employ spectral analysis tools to show that system (40) has a sequence of eigenvalues which tends to infinity and is bounded from below by $-\lambda^2/4$.

Making usage of the characteristic equation of the equation in (40), i.e.,

$$r^4 + \lambda r^2 - \sigma = 0$$

in view of the notations in (18) we distinguish several cases as follows.

Case I: $\sigma > 0$. In this case, the general solution of the equation in (40) is given by

$$\phi(x) = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x) + C_3 \cos(\beta x) + C_4 \sin(\beta x),$$

where C_i are real constants, $i = 1, 4$. Imposing the boundary conditions at $x = 0$ we easily obtain that $C_2 = C_4 = 0$. The boundary conditions at $x = L$ provide a nontrivial solution ϕ if $\sinh(\alpha L) \sin(\beta L) = 0$. Since $\alpha > 0$ this is equivalent to the compatibility condition

$$\sin(\beta L) = 0.$$

Then, we get a sequence $\{\beta_n\}_{n \geq 1}$ of positive solutions, $\beta_n = n\pi/L$. In view of (18) we obtain that the sequence of positive simple eigenvalues is given by

$$\sigma_n = \left(\frac{n\pi}{L}\right)^4 - \lambda \left(\frac{n\pi}{L}\right)^2, \quad n \geq \left\lfloor \frac{L\sqrt{\lambda}}{\pi} \right\rfloor + 1,$$

where $\lfloor \cdot \rfloor$ is the floor function. The corresponding eigenfunctions are

$$\phi_n(x) = C_1 \cos(\beta_n x), \quad C_1 \neq 0.$$

Case II: $\sigma = 0$. The general solution for the equation in (40) is

$$\phi(x) = C_1 + C_2 x + C_3 \cos(\sqrt{\lambda} x) + C_4 \sin(\sqrt{\lambda} x).$$

From the boundary conditions at $x = 0$ we deduce that $C_2 = C_4 = 0$. According to the boundary conditions at $x = L$ we produce the following alternatives.

- (1) If $\sin(\sqrt{\lambda} L) = 0$ then $\sigma = 0$ is an eigenvalue with multiplicity 2 and the eigenfunctions are

$$\phi_0(x) = C_1 + C_3 \cos(\sqrt{\lambda} x), \quad C_1^2 + C_3^2 \neq 0.$$

- (2) If $\sin(\sqrt{\lambda} L) \neq 0$ then $\sigma = 0$ is a simple eigenvalue and the eigenfunctions are constant functions, i.e.,

$$\phi_0(x) = C, \quad C \neq 0.$$

Case III: $-\lambda^2/4 < \sigma < 0$. The general solution of the equation in (40) is

$$\phi(x) = C_1 \cos(\gamma x) + C_2 \sin(\gamma x) + C_3 \cos(\beta x) + C_4 \sin(\beta x).$$

The boundary conditions at $x = 0$ lead to $C_2 = C_4 = 0$. Defining the quantity $\delta := \sin(\beta L) \sin(\gamma L)$, from the boundary conditions at $x = L$ we obtain that ϕ is an eigenfunction if and only if $\delta = 0$. We distinguish the following cases for $\delta = 0$.

- (1) *The case $\sin(\beta L) = \sin(\gamma L) = 0$.* We obtain two sequences of solutions $\beta_n = n\pi/L$ and $\gamma_m = m\pi/L$ with $n \neq m$ (since $\beta \neq \gamma$), $n, m \geq 1$. From (19) this is equivalent to $\lambda \in \mathcal{N}_1$ and in this case the finite set of negative eigenvalues is given by

$$\sigma_{n,m}(x) = -\beta_n^2 \gamma_m^2 = -\frac{n^2 m^2 \pi^4}{L^4}; \quad 1 \leq m < n, \quad nm < \frac{\lambda L^2}{2\pi^2}.$$

The corresponding eigenfunctions are

$$\phi_{n,m}(x) = C_1 \cos(\beta_n x) + C_3 \cos(\beta_m x), \quad C_1^2 + C_3^2 \neq 0.$$

- (2) *The case $\delta = 0$, such that $\sin^2(\beta L) + \sin^2(\gamma L) > 0$, i.e. $\lambda \notin \mathcal{N}_1$.* Then we obtain the eigenvalues, i.e.

$$\sigma_n = \left(\frac{n\pi}{L}\right)^4 - \lambda \left(\frac{n\pi}{L}\right)^2, \quad 1 \leq n < \frac{L\sqrt{\lambda}}{\pi},$$

with the corresponding eigenfunctions

$$\phi_n(x) = C \cos\left(\frac{n\pi x}{L}\right), \quad C \neq 0.$$

From the spectral analysis developed above it is easy to check the following lemma that will play an important role in the proof of Theorem 1.2.

Lemma 3.1. *Let $\lambda > 0$ and (σ, ϕ) be an eigenpair of system (40). The following holds:*

- (1) *If $\sigma > 0$ then*

$$\phi(L) \neq 0 \text{ and } \lambda\phi(L) + \phi_{xx}(L) \neq 0.$$

- (2) *If $\sigma = 0$ and $\lambda \in \mathcal{N}_2$ then $\phi(L)$ and $\lambda\phi(L) + \phi_{xx}(L)$ cannot vanish simultaneously.*

- (3) *If $\sigma = 0$ and $\lambda \notin \mathcal{N}_2$ then $\phi(L) \neq 0$ and $\lambda\phi(L) + \phi_{xx}(L) \neq 0$.*

- (4) *If $\sigma < 0$ and $\lambda \notin \mathcal{N}_1$ then*

$$\phi(L) \neq 0 \text{ and } \lambda\phi(L) + \phi_{xx}(L) \neq 0.$$

- (5) *If $\sigma < 0$ and $\lambda \in \mathcal{N}_1$ then $\phi(L)$ and $\lambda\phi(L) + \phi_{xx}(L)$ cannot vanish simultaneously.*

3.2. Preliminaries II. Secondly we analyze the following eigenvalue problem

$$(41) \quad \begin{cases} \lambda\phi_{xx} + \phi_{xxxx} = \sigma\phi, & x \in (0, L), \\ \phi(0) = \phi_{xx}(0) = 0, \\ \phi_x(L) = \phi_{xxx}(L) = 0. \end{cases}$$

Again, the spectrum of (41) is pure discrete, bounded from below by $-\lambda^2/4$ and tends to infinity. Making use of the notations (18) we distinguish the following cases.

Case I: $\sigma > 0$. The solution of the equation in (40) is

$$\phi(x) = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x) + C_3 \cos(\beta x) + C_4 \sin(\beta x).$$

From the boundary conditions at $x = 0$ we easily obtain that $C_1 = C_3 = 0$. The conditions at $x = L$ say that ϕ is an eigenfunction under the constraint

$$\cos(\beta L) = 0.$$

We get a sequence $\{\beta_n\}_{n \geq 0}$ of positive solutions, $\beta_n = (2n+1)\pi/2L$. In view of (18)-(19) we obtain the sequence of simple eigenvalues $\sigma_n = \beta_n^2(\beta_n^2 - \lambda)$, i.e.

$$\sigma_n = \frac{(2n+1)^2 \pi^2}{4L^2} \left(\frac{(2n+1)^2 \pi^2}{4L^2} - \lambda \right), \quad n \geq \max \left\{ 0, \left\lceil \frac{1}{2} \left(\frac{2L\sqrt{\lambda}}{\pi} - 1 \right) \right\rceil + 1 \right\},$$

with the corresponding family of eigenfunctions

$$\phi_n(x) = C_2 \sin(\beta_n x), \quad C_2 \neq 0.$$

Case II: $\sigma = 0$. The general solution for the equation in (40) is

$$\phi(x) = C_1 + C_2 x + C_3 \cos(\sqrt{\lambda} x) + C_4 \sin(\sqrt{\lambda} x).$$

From the boundary conditions at $x = 0$ we deduce that $C_1 = C_3 = 0$. Then, the boundary conditions at $x = L$ produce the following cases.

- (1) If $\cos(\sqrt{\lambda} L) \neq 0$, which is equivalent to $\lambda \notin \mathcal{N}_3$, then $\sigma = 0$ is not an eigenvalue.
- (2) On the contrary, if $\cos(\sqrt{\lambda} L) = 0$, which is equivalent to $\lambda \in \mathcal{N}_3$, then $\sigma = 0$ is a simple eigenvalue with the corresponding eigenfunctions

$$\phi_0(x) = C_4 \sin(\sqrt{\lambda} x), \quad C_4 \neq 0.$$

Case III: $-\lambda^2/4 < \sigma < 0$. The general solution of the equation in (41) is

$$\phi(x) = C_1 \cos(\gamma x) + C_2 \sin(\gamma x) + C_3 \cos(\beta x) + C_4 \sin(\beta x).$$

The boundary conditions at $x = 0$ give $C_1 = C_3 = 0$. Then, from the conditions at $x = L$ we obtain that ϕ is an eigenfunction if and only if

$$\cos(\beta L) \cos(\gamma L) = 0.$$

We distinguish the following cases.

- (1) *The case* $\cos(\beta L) = \cos(\gamma L) = 0$. We obtain $\beta_n = (2n+1)\pi/2L$ and $\gamma_m = (2m+1)\pi/2L$, with $n \neq m$ (since $\beta \neq \gamma$). From (18)-(19) this is equivalent to $\lambda \in \mathcal{N}_{odd}$. In this case the eigenvalues have multiplicity 2 and they are given by

$$\sigma_{n,m}(x) = -\beta_n^2 \beta_m^2; \quad 0 \leq m < n, \quad (2n+1)(2m+1) \leq \frac{2\lambda L^2}{\pi^2}.$$

The corresponding eigenfunctions are

$$\phi_{n,m}(x) = C_2 \sin(\beta_n x) + C_4 \sin(\beta_m x), \quad C_1^2 + C_3^2 \neq 0.$$

- (2) *The case* $\cos^2(\beta L) + \cos^2(\gamma L) > 0$, i.e. $\lambda \notin \mathcal{N}_{odd}$. We obtain a finite number of simple eigenvalues such as

$$\sigma_n = \frac{(2n+1)^2 \pi^2}{4L^2} \left(\frac{(2n+1)^2 \pi^2}{4L^2} - \lambda \right), \quad 0 \leq n \leq \max \left\{ \left\lceil \frac{1}{2} \left(\frac{2L\sqrt{\lambda}}{\pi} - 1 \right) \right\rceil, 0 \right\}.$$

The corresponding eigenfunctions are given by

$$\phi_n(x) = C \sin \left(\frac{(2n+1)\pi x}{2L} \right), \quad C \neq 0.$$

Combining the spectral results of this section we conclude

Lemma 3.2. *Let $\lambda > 0$ and (σ, ϕ) be an eigenpair of system (41).*

(1) *If $\sigma > 0$ then*

$$\phi(L) \neq 0 \text{ and } \lambda\phi(L) + \phi_{xx}(L) \neq 0.$$

(2) *If $\lambda \notin \mathcal{N}_3$ then $\sigma = 0$ is not an eigenvalue.*

(3) *If $\lambda \in \mathcal{N}_3$ then $\sigma = 0$ is an eigenvalue and*

$$\phi(L) = \lambda\phi(L) + \phi_{xx}(L) = 0.$$

(4) *If $\sigma < 0$ and $\lambda \in \mathcal{N}_{\text{odd}}$ then $\phi(L)$ and $\lambda\phi(L) + \phi_{xx}(L)$ cannot vanish simultaneously.*

(5) *If $\sigma < 0$ and $\lambda \notin \mathcal{N}_{\text{odd}}$ then*

$$\phi(L) \neq 0 \text{ and } \lambda\phi(L) + \phi_{xx}(L) \neq 0.$$

3.3. Spectral analysis. In this section we aim to discuss some general properties of the following spectral problem

$$(42) \quad \begin{cases} \lambda\phi_{xx}^k + \phi_{xxxx}^k = \sigma\phi^k, & x \in (0, L), \\ \phi^i(0) = \phi^j(0), & i, j \in \{1, \dots, N\}, \\ \phi_x^k(L) = \phi_{xxx}^k(L) = 0, & k \in \{1, \dots, N\}, \\ \sum_{k=1}^N \phi_x^k(0) = 0, \\ \phi_{xx}^i(0) = \phi_{xx}^j(0), & i, j \in \{1, \dots, N\}, \\ \sum_{k=1}^N \phi_{xxx}^k(0) = 0, \end{cases}$$

which governs our control system (2)-(II). This is equivalent to study the spectral properties of the fourth order operator

$$A : D(A) \subset L^2(\Gamma) \rightarrow L^2(\Gamma)$$

given by

$$(43) \quad \begin{cases} A\phi^k = \lambda\phi_{xx}^k + \phi_{xxxx}^k, & k \in \{1, \dots, N\} \\ D(A) = \left\{ \begin{array}{l} \phi = (\phi^k)_{k=1, N} \in H^4(\Gamma) \mid \phi_x^k(L) = \phi_{xxx}^k(L) = 0, \quad k \in \{1, \dots, N\}, \\ \phi^i(0) = \phi^j(0), \quad \phi_{xx}^i(0) = \phi_{xx}^j(0), \quad i, j \in \{1, \dots, N\}, \\ \sum_{k=1}^N \phi_x^k(0) = 0, \quad \sum_{k=1}^N \phi_{xxx}^k(0) = 0, \end{array} \right\}. \end{cases}$$

Similar to the operator induced by the model (2)-(I) we obtain

Proposition 3.3. *For any $\mu > \lambda^2/4$ the operator*

$$A + \mu I : D(A) \subset L^2(\Gamma) \rightarrow L^2(\Gamma),$$

is a non-negative self-adjoint operator with compact inverse. In particular, it has a pure discrete spectrum consisted by a sequence of nonnegative eigenvalues $\{\sigma_{\mu, l}\}_{l \in \mathbb{N}}$ satisfying $\lim_{l \rightarrow \infty} \sigma_{\mu, l} = \infty$. Moreover, up to a normalization, the corresponding eigenfunctions $\{\phi_{\mu, l}\}_{l \in \mathbb{N}}$ form an orthonormal basis of $L^2(\Gamma)$.

Moreover,

Proposition 3.4. *The spectrum $\{\sigma_l\}_{l \in \mathbb{N}}$ of problem (42) verifies*

$$(44) \quad -\frac{\lambda^2}{4} < \sigma_l, \quad \forall l \in \mathbb{N}, \quad \sigma_l \rightarrow \infty, \text{ as } l \rightarrow \infty.$$

The details of the proof of Propositions 3.3 and 3.4 are quite similar as in the model (2)-(I), therefore they will be omitted here.

3.4. Qualitative properties of the eigenvalues. In this section we apply the preliminary results shown in subsections (3.1) and (3.2) to obtain useful properties of the eigenvalues of (42) which will play a crucial role for proving the null-controllability of problem (2)-(II). In particular we obtain the asymptotic behavior of the eigenvalues of system (42).

For any fixed $\lambda > 0$ let us firstly consider the following two eigenvalue problems which have been already analyzed in subsections (3.1) and (3.2):

$$(45) \quad E_1 : \begin{cases} \lambda \Psi_{xx} + \Psi_{xxxx} = \sigma \Psi, & x \in (0, L), \\ \Psi_x(0) = 0, \quad \Psi_{xxx}(0) = 0, \\ \Psi_x(L) = \Psi_{xxx}(L) = 0, \end{cases}$$

and

$$(46) \quad E_2 : \begin{cases} \lambda \Phi_{xx} + \Phi_{xxxx} = \sigma \Phi, & x \in (0, L), \\ \Phi(0) = 0, \quad \Phi_{xx}(0) = 0, \\ \Phi_x(L) = \Phi_{xxx}(L) = 0, \end{cases}$$

respectively. Recall that Lemma 3.1 applies for (45) whereas Lemma 3.2 applies for (46). Coming back to our spectral problem (42), we introduce the functions

$$(47) \quad S := \sum_{k=1}^N \phi^k,$$

$$(48) \quad D^k := \phi^k - \frac{S}{N}, \quad k \in \{1, \dots, N\}.$$

The motivation for analyzing systems (45) and (46) is due to the fact that S verifies (45) whereas D^k satisfies (46) for all $k \in \{1, \dots, N\}$.

Next we state and prove some preliminary lemmas.

Lemma 3.5. *For any $\lambda > 0$ the eigenvalue problems (45) and (46) have no any common positive eigenvalues. The value $\sigma = 0$ is a common eigenvalue if and only if λ belongs to \mathcal{N}_3 . Moreover, problems (45) and (46) have no common negative eigenvalues if and only if $\lambda \notin \mathcal{N}_4$.*

Proof. Assume that $\sigma > 0$ is a common eigenvalue for (45) and (46). Then, according to the precise analysis in subsections 3.1 and 3.2 we must necessary have

$$\sin(\beta L) = \cos(\beta L),$$

which never may happen. Again, in view of the subsections above we obtain that $\sigma = 0$ is an eigenvalue for (45) but it cannot be an eigenvalue for (46) unless $\lambda \in \mathcal{N}_3$. Moreover, if some $\sigma < 0$ was a common eigenvalue we should have

$$\sin(\beta L) \sin(\gamma L) = \cos(\beta L) \cos(\gamma L) = 0,$$

which is equivalent to the alternatives

$$\sin(\beta L) = \cos(\gamma L) = 0 \text{ or } \sin(\gamma L) = \cos(\beta L) = 0.$$

This is impossible unless $\lambda \in \mathcal{N}_4$. □

In consequence we have the following partition for the set of eigenvalues of system (9).

Lemma 3.6. *If $\lambda \notin \mathcal{N}_{mixt}$ then*

$$(49) \quad \sigma_p(A) = \sigma_p(E_1) \cup \sigma_p(E_2); \quad \sigma_p(E_1) \cap \sigma_p(E_2) = \emptyset,$$

where $\sigma_p(A)$, $\sigma_p(E_1)$ and $\sigma_p(E_2)$ denote the set of eigenvalues for the spectral problems (42), (45) and (46), respectively.

Proof. “Only if” implication. Assume $(\sigma, \phi = (\phi^k)_{k=1,N})$ is an eigenpair of (42). Then σ verifies (45) for $\Psi = S$ in (47). In addition, σ verifies (46) for any $\Phi = D^k$ in (48). If $S \neq 0$ then (σ, S) is an eigenpair for (45). Otherwise, if $S = 0$, according to Lemma 3.5 we have $D^k = \phi^k$ for all $k \in \{1, \dots, N\}$. Consequently, there exists $k_0 \in \{1, \dots, N\}$ such that $\phi^{k_0} \neq 0$ and therefore (σ, ϕ^{k_0}) is an eigenpair for (46).

“If” implication. Let (σ, Ψ) be an eigenpair for (45). Then $(\sigma, \phi = (\Psi, \dots, \Psi, \Psi))$ is an eigenpair for (42). Let (σ, Φ) be an eigenpair for (46). Then $(\sigma, \phi = (0, \dots, 0, -\Phi, \Phi))$ is an eigenpair for (42), which completes the proof of Lemma 3.6. \square

The previous results allow us to conclude the following lemma.

Lemma 3.7. *For any $\lambda \notin \mathcal{N}_{mixt} \cup \mathcal{N}_{even}$ any eigenfunction $\phi = (\phi^k)_{k=1,N}$ of A in (43) satisfies $\phi^k(L) \neq 0$ or $\lambda\phi^k(L) + \phi_{xx}^k(L) \neq 0$ for at least two indexes $k \in \{1, \dots, N\}$.*

Proof. With the same notations as above we have that S and D^k , $k \in \{1, \dots, N\}$, satisfy (45) and (46). We distinguish two cases as follows.

The case $S \neq 0$. Using Lemma 3.6 we must have $D^k \equiv 0$, for all $k \in \{1, \dots, N\}$. This means that $\phi = 1/N(S, \dots, S)$ where S is the eigenfunction of problem (45). So, from Lemma 3.1 it holds that $S(L) \neq 0$ or $\lambda S(L) + S_{xx}(L) \neq 0$. This gives the desired result.

The case $S \equiv 0$. In this case we have $D^k = \phi^k$ for all $k \in \{1, \dots, N\}$. Assume that for at least $N - 1$ indexes $k \in \{1, \dots, N\}$ we have $\phi^k(L) = 0$ or $\lambda\phi^k(L) + \phi_{xx}^k(L) = 0$. Then we must have $D^k(L) = 0$ for all $k \in \{1, \dots, N\}$ or $\lambda D^k(L) + D_{xx}^k(L) = 0$ for all $k \in \{1, \dots, N\}$. On the other hand, we know that there exists $k_0 \in \{1, \dots, N\}$ such that $\phi^{k_0} \neq 0$. This implies that D^{k_0} is an eigenfunction for (46). Since $\lambda \notin \mathcal{N}_{even}$, applying Lemma 3.1 we must have $D_{xx}^{k_0}(L) \neq 0$ and $\lambda\phi^{k_0}(L) + \phi_{xx}^{k_0}(L) \neq 0$, which leads to a contradiction.

Therefore, the proof is finished. \square

Lemma 3.8. *The eigenvalues of problem (42) satisfy*

$$\sigma_n = Kn^4 + O(n^3), \quad n \rightarrow \infty,$$

for some uniform constant $K > 0$.

The proof of Lemma 3.8 is a direct consequence of the analysis done in subsections 3.1 and 3.2.

3.5. Controllability problem. The control problem (2)-(II) is reduced to solve the following moment problem. Similarly as in Lemma 2.10 we can show

Lemma 3.9. *Let $\{\phi_l\}_{l \in \mathbb{N}}$ be the orthonormal basis of $L^2(\Gamma)$ formed by the eigenfunctions of A corresponding to the eigenvalues $\{\sigma_l\}_{l \in \mathbb{N}}$. Then system (2) is null-controllable if for any initial data $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$,*

$$y_{0,l} = \sum_{l \in \mathbb{N}} y_{0,l} \phi_l,$$

and any time $T > 0$, there exist controls $u = (a^k, b^k)_{k=1, N} \in H^1(\Gamma)$ such that

$$\begin{aligned} y_{0,l}^k e^{-T\sigma_l} &= \sum_{k=1}^N (\lambda \phi_l^k(L) + \phi_{l,xx}^k(L)) \int_0^T a^k(T-t) e^{-t\sigma_l} dt \\ &\quad + \sum_{k=1}^N \phi_l^k(L) \int_0^T b^k(T-t) e^{-t\sigma_l} dt, \quad \forall l \in \mathbb{N}. \end{aligned}$$

3.6. Proof of Theorem 1.2. As a consequence of Lemmas 3.7 and 3.8 we can again apply the strategy of Fattorini-Russell [12] to obtain the null-controllability results for system (2)-(II). The proof is very similar to the proof of Theorem 1.1, therefore we avoid the details here.

In view of the detailed proofs of the main controllability results we remark that the results in Theorems 1.1 and 1.2 are optimal. More precisely, in general the systems (2)-(I) and (2)-(II) are not null-controllable when $\lambda \in \mathcal{N}_{even} \cup \mathcal{N}_{odd}$ or $\lambda \in \mathcal{N}_{even} \cup \mathcal{N}_{mixt}$ not even with controls acting on the all components. Moreover, the number of the active control components in Theorems 1.1 and 1.2 is also optimal.

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